

**CONSEQUENCES OF THE EXISTENCE OF  
AUSLANDER-REITEN TRIANGLES WITH APPLICATIONS  
TO PERFECT COMPLEXES FOR SELF-INJECTIVE  
ALGEBRAS**

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**ABSTRACT.** In a  $k$ -linear triangulated category (where  $k$  is a field) we show that the existence of Auslander-Reiten triangles implies that objects are determined, up to shift, by knowing dimensions of homomorphisms between them. In most cases the objects themselves are distinguished by this information, a conclusion which was also reached under slightly different hypotheses in a theorem of Jensen, Su and Zimmermann. The approach is to consider bilinear forms on Grothendieck groups which are analogous to the Green ring of a finite group.

We specialize to the category of perfect complexes for a self-injective algebra, for which the Auslander-Reiten quiver has a known shape. We characterize the position in the quiver of many kinds of perfect complexes, including those of lengths 1, 2 and 3, rigid complexes and truncated projective resolutions. We describe completely the quiver components which contain projective modules. We obtain relationships between the homology of complexes at different places in the quiver, deducing that every self-injective algebra of radical length at least 3 has indecomposable perfect complexes with arbitrarily large homology in any given degree. We find also that homology stabilizes away from the rim of the quiver. We show that when the algebra is symmetric, one of the forms considered earlier is Hermitian, and this allows us to compute its values knowing them only on objects on the rim of the quiver.

## 1. INTRODUCTION

We will consider triangulated categories which are  $k$ -linear for some algebraically closed field  $k$  and for which the Krull-Schmidt theorem holds, meaning that every object is a finite direct sum of indecomposable objects, and endomorphism rings of indecomposable objects are local. We will also assume that Auslander-Reiten triangles exist. An important example of such a category for us is the category  $D^b(\Lambda\text{-proj})$  of perfect complexes of  $\Lambda$ -modules with homotopy classes of maps, where  $\Lambda$  is a self-injective finite

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dimensional  $k$ -algebra. Other categories of interest include cluster categories, the stable module category of  $\Lambda$ -modules when  $\Lambda$  is self-injective, and  $D^b(\Lambda\text{-mod})$  when  $\Lambda$  is an algebra of finite global dimension.

We approach such triangulated categories from two angles. We first consider the role of the Auslander-Reiten triangles in the context of two related Grothendieck groups, analogous to the Green ring in group representation theory. These Grothendieck groups carry bilinear forms determined by the dimensions of homomorphisms between complexes. We show that the bilinear forms constructed in this way are non-degenerate in a certain sense (Corollary 3.2). We also show that for the second Grothendieck group we construct, in the case of  $D^b(\Lambda\text{-proj})$  when  $\Lambda$  is symmetric, the form is Hermitian and the Auslander-Reiten triangles provide a set of elements dual to the basis determined by indecomposable complexes (Theorem 4.2). This development is very much in the spirit of [6].

We next pay attention to  $D^b(\Lambda\text{-proj})$  when  $\Lambda$  is a self-injective algebra. To explain the notation,  $\Lambda\text{-proj}$  denotes the full subcategory of all  $\Lambda$ -modules whose objects are finitely generated projective, and  $D^b(\Lambda\text{-proj})$  may be taken to be the category whose objects are finite complexes of finitely generated projective modules – the *perfect* complexes – with homotopy classes of chain maps as the morphisms. Such perfect complexes appear in many places in the representation theory of algebras and elsewhere, with many applications. They arise as tilting complexes in Rickard’s theory [11] of derived equivalences of rings. They are the compact objects in the bounded derived category of a ring [11, 6.3]. In the context of free group actions on topological spaces, a CW complex on which a group acts freely has a chain complex which is a perfect complex of modules for the group algebra (see [1] for a notable contribution in this area). In the theory of subgroup complexes of a finite group, the chain complex of the poset of non-identity  $p$ -subgroups, over a  $p$ -local ring, is homotopy equivalent to a perfect complex of modules for the group algebra [12]. These are just a few examples.

When  $\Lambda$  is self-injective it is known that  $D^b(\Lambda\text{-proj})$  has Auslander-Reiten triangles [7], and furthermore that the Auslander-Reiten quiver has shape  $\mathbb{Z}A_\infty$  [13]. We exploit this to study the position of perfect complexes of various kinds in their Auslander-Reiten quiver component. We completely describe all the complexes in quiver components which contain projective modules (Theorem 6.6). For complexes of length 2 or 3 we show that they all lie within distance 2 of the rim, and we describe which ones lie on the rim, which lie at distance 1 and which lie at distance 2 (Corollary 6.8 and Proposition 7.3). In general, complexes at distance  $n$  from the rim must have length at least  $n+1$  (Corollary 6.10). We prove this by studying the homology modules of complexes in the interior of the quiver component, showing that they have composition factors which are the union of the composition factors of the homology modules lying on the rim of the corresponding ‘wing’

in the quiver (Proposition 6.4). We show also that homology modules stabilize, in a certain sense, away from the rim of the quiver (Theorem 6.5), so that to each perfect complex there is associated a *stabilization module*.

Some of our conclusions can be expressed without the formalism of the triangulated categories. We prove that when  $\Lambda$  is a symmetric algebra of radical length at least 3 there are always indecomposable complexes with zero homology of arbitrarily large dimension (Theorem 6.13). We also show that for each Auslander-Reiten *sequence* of  $\Lambda$  modules there is a three-term perfect complex which has the three modules in the sequence as its homology groups (Corollary 7.2). Given three modules appearing in a short exact sequence it is not, in general, possible to realize them as homology of a perfect complex of length 3, so the fact that it can be done for Auslander-Reiten sequences is a positive result. The construction of the three-term complex also provides an algorithm (which works for an arbitrary finite dimensional algebra) for constructing Auslander-Reiten sequences on a computer.

Here is the organization of this paper in greater detail. In Section 2 we present the basic property of Auslander-Reiten triangles with respect to homomorphisms on which the subsequent arguments depend (Lemma 2.1). The setting is that of a Krull-Schmidt triangulated category with at least one Auslander-Reiten triangle. In Section 3 we define the first Grothendieck group in the same generality, and show that the defining property of the Auslander-Reiten triangles implies a non-singularity property of the bilinear form given by dimensions of homomorphisms (Corollary 3.2). We deduce that in many situations objects are determined by knowing the dimensions of homomorphisms to other objects (Corollary 3.3). This is something which is analogous to the approach of Benson and Parker [6] in their study of the Green ring of group representations and which has been proven for triangulated categories with slightly different conditions by Jensen, Su and Zimmermann [9].

In Section 4 we modify the Grothendieck group so that it becomes a  $\mathbb{Q}(t)$ -vector space on which  $t$  acts as the shift operator. With respect to the field automorphism determined by  $t \mapsto t^{-1}$  the earlier bilinear form on the first Grothendieck group induces a sesquilinear bilinear form on the modified group, which in case our triangulated category is the category of perfect complexes for a symmetric algebra turns out to be Hermitian (Theorem 4.2). In Section 5 we work with perfect complexes for a symmetric algebra, and use the Hermitian property of the form to calculate its values, showing that they only depend on values of complexes on the rim of the Auslander-Reiten quiver (Theorem 5.1).

The contents of Sections 6 and 7 have already been largely described, having to do with the position of complexes in the quiver and properties of their homology. Apart from the results so far mentioned we point out a characterization of complexes of length  $n + 1$  which lie at the maximal distance  $n$  from the rim of the quiver (Corollary 6.10). These complexes only appear in quiver components containing projective modules, and they

are completely described. We also show that truncated projective resolutions lie on the rim, except in one situation (Proposition 6.12). In Section 8 we show that rigid complexes lie on the rim of the quiver (Theorem 8.2).

Auslander-Reiten triangles were introduced by Happel in the 1980s, and his book [7] is a very good place to read about them and find background material on triangulated categories. The definition and first properties of these triangles can be found there. Happel also shows that in the bounded derived category an Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  exists if and only if  $Z$  is isomorphic to a perfect complex, which happens if and only if  $X$  is isomorphic to a finite complex of finitely generated injective modules. From this it follows that Auslander-Reiten triangles exist in  $D^b(\Lambda\text{-proj})$  when  $\Lambda$  is self-injective. He describes a construction of these triangles in the bounded derived category  $D^b(\Lambda\text{-mod})$ , which also works for the perfect complexes  $D^b(\Lambda\text{-proj})$  and which we will use all the time. The construction uses the Nakayama functor  $\nu : \Lambda\text{-mod} \rightarrow \Lambda\text{-mod}$  which is described in [2]. This functor is extended to complexes by applying  $\nu$  to each term and each morphism in the complex, and the resulting functor  $D^b(\Lambda\text{-proj}) \rightarrow D^b(\Lambda\text{-inj})$  is also denoted  $\nu$ .

## 2. THE BASIC LEMMA

We assume that  $\mathcal{C}$  is a Krull-Schmidt  $k$ -linear triangulated category which is Hom-finite, that is, the homomorphism spaces between objects are finite dimensional over a field  $k$ . For convenience we assume that  $k$  is algebraically closed.

Many of our results depend upon the following observation, which follows directly from the definition of an Auslander-Reiten triangle.

**Lemma 2.1.** *Let  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  be an Auslander-Reiten triangle in a  $k$ -linear triangulated category  $\mathcal{C}$  and let  $W$  be an indecomposable object of  $\mathcal{C}$ . Consider the long exact sequence obtained by applying  $\text{Hom}_{\mathcal{C}}(W, -)$  to the triangle.*

- (1) *If  $W \not\cong Z[r]$  for any  $r$  then the long exact sequence is a splice of short exact sequences*

$$0 \rightarrow \text{Hom}(W, X[n]) \rightarrow \text{Hom}(W, Y[n]) \rightarrow \text{Hom}(W, Z[n]) \rightarrow 0.$$

- (2) *If  $W \cong Z[r]$  for some  $r$  and  $Z \not\cong Z[1]$  the long exact sequence is still the splice of short exact sequences as above, except that the sequences for  $n = r$  and  $r + 1$  combine to give a 6-term exact sequence whose middle connecting homomorphism  $\delta$  has rank 1:*

$$\begin{aligned} 0 &\rightarrow \text{Hom}(W, X[r]) \rightarrow \text{Hom}(W, Y[r]) \rightarrow \text{Hom}(W, Z[r]) \\ &\xrightarrow{\delta} \text{Hom}(W, X[r+1]) \rightarrow \text{Hom}(W, Y[r+1]) \rightarrow \text{Hom}(W, Z[r+1]) \\ &\rightarrow 0. \end{aligned}$$

(3) If  $W \cong Z \cong Z[1]$  the long exact sequence becomes a repeating exact sequence with three terms:

$$\text{Hom}(W, X) \rightarrow \text{Hom}(W, Y) \rightarrow \text{Hom}(W, Z) \xrightarrow{\delta} \text{Hom}(W, X) \rightarrow \dots$$

where the connecting homomorphism  $\delta$  has rank 1.

The dual result on applying  $\text{Hom}_{\mathcal{C}}(-, W)$  to the triangle also holds.

*Proof.* In the long exact sequence

$$\begin{array}{ccccccc} & & \dots & & \text{Hom}(W, Z[r-1]) & \curvearrowright \\ & & \xrightarrow{\beta_{r-1}} & & & & \\ \curvearrowleft & \text{Hom}(W, X[r]) & \longrightarrow & \text{Hom}(W, Y[r]) & \xrightarrow{\beta_r} & \text{Hom}(W, Z[r]) & \curvearrowright \\ \curvearrowleft & & & & & & \\ & & \text{Hom}(W, X[r+1]) & \rightarrow & \text{Hom}(W, Y[r+1]) & \xrightarrow{\beta_{r+1}} & \text{Hom}(W, Z[r+1]) \\ \curvearrowleft & & & & & & \\ & & \text{Hom}(W, X[r+2]) & \rightarrow & & \dots & \end{array}$$

the map  $\beta_r$  is surjective unless  $W \cong Z[r]$ . Thus if  $W \not\cong Z[r]$  for all  $r$ , the connecting homomorphisms are all zero, which forces the long sequence to be a splice of short exact sequences. In the case where  $W = Z[r]$  for some  $r$ , the map  $\beta_r$  has image  $\text{Rad End}(W)$  and cokernel of dimension 1, from the lifting property of an Auslander-Reiten triangle. If  $Z \not\cong Z[1]$  it follows that  $Z[s] \not\cong Z[s+1]$  for any  $s$ , so  $Z[r-1] \not\cong W \not\cong Z[r+1]$  and both  $\beta_{r-1}$  and  $\beta_{r+1}$  are surjective, giving a six-term exact sequence. When  $Z \cong Z[1]$  then all shifts of  $Z$  are isomorphic, as are the shifts of  $X$  and of  $Y$ . Evidently the long exact sequence becomes the three-term sequence shown.  $\square$

We deduce an immediate consequence of this lemma for the homology of complexes of  $\Lambda$ -modules. Observe that the functor which sends such a complex  $C$  to its  $n$ th homology group  $H_n(C)$  is representable, since  $H_n(C) = \text{Hom}_{D^b(\Lambda)}(\Lambda[n], C)$ . Here  $\Lambda[n]$  denotes the complex which is zero in every position except for position  $n$ , where it is  $\Lambda$ .

**Corollary 2.2.** *Let  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  be an Auslander-Reiten triangle in  $D^b(\Lambda\text{-mod})$  or  $D^b(\Lambda\text{-proj})$  where  $\Lambda$  is a finite dimensional  $k$ -algebra.. The long exact sequence of homology groups of this triangle breaks up as the splice of short exact sequences of the form  $0 \rightarrow H_n(X) \rightarrow H_n(Y) \rightarrow H_n(Z) \rightarrow 0$  except when  $Z$  is a shift of a projective module.*

*Proof.* This is immediate from Proposition 2.1 using the fact that homology is representable as  $\text{Hom}(\Lambda, -)$ , so that the long exact sequence is a splice of short exact sequences if  $Z$  is not a projective module.  $\square$

### 3. THE FIRST GROTHENDIECK GROUP

Again let  $\mathcal{C}$  be a Krull-Schmidt  $k$ -linear triangulated category which is Hom-finite, and where  $k$  is algebraically closed. We define  $A(\mathcal{C})$  to be the free abelian group with the isomorphism classes  $[C]$  of indecomposable objects  $C$  as basis. If  $C = C_1 \oplus \cdots \oplus C_n$  we put  $[C] = [C_1] + \cdots + [C_n]$ . We define a bilinear form

$$\langle , \rangle : A(\mathcal{C}) \times A(\mathcal{C}) \rightarrow \mathbb{Z}$$

by  $\langle [C], [D] \rangle := \dim \text{Hom}_{\mathcal{C}}(C, D)$ . If  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is a triangle in  $\mathcal{C}$  we put  $\hat{Z} := [Z] + [X] - [Y]$  in  $A(\mathcal{C})$ .

**Proposition 3.1.** *Let  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  be an Auslander-Reiten triangle in  $\mathcal{C}$  and let  $W$  be an indecomposable object. If  $Z \not\cong Z[1]$  then*

$$\langle W, \hat{Z} \rangle = \begin{cases} 1 & \text{if } W \cong Z \text{ or } Z[-1] \\ 0 & \text{otherwise.} \end{cases}$$

If  $Z \cong Z[1]$  then

$$\langle W, \hat{Z} \rangle = \begin{cases} 2 & \text{if } W \cong Z \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Observe that  $\langle W, \hat{Z} \rangle$  is the alternating sum of the dimensions of the vector spaces in the (not necessarily exact) sequence

$$0 \rightarrow \text{Hom}(W, X) \xrightarrow{\alpha_0} \text{Hom}(W, Y) \xrightarrow{\beta_0} \text{Hom}(W, Z) \rightarrow 0.$$

From Lemma 2.1 this is a short exact sequence unless  $W \cong Z$  or  $Z[-1]$ , and hence apart from these cases  $\langle W, \hat{Z} \rangle = 0$ . Assuming that  $Z \not\cong Z[1]$ , if  $W \cong Z$  then the sequence is exact except at the right, where  $\text{Coker } \beta_0$  has dimension 1, and if  $W \cong Z[-1]$  then the sequence is exact except at the left, where  $\text{Ker } \alpha_0$  has dimension 1. Thus in these cases  $\langle W, \hat{Z} \rangle = 1$ . Finally when  $W \cong Z \cong Z[1]$  we see from Lemma 2.1 that both  $\text{Ker } \alpha_0$  and  $\text{Coker } \beta_0$  have dimension 1, so that  $\langle W, \hat{Z} \rangle = 2$ .  $\square$

**Corollary 3.2.** *Suppose that  $\mathcal{C}$  has Auslander-Reiten triangles. Let*

$$\phi : A(\mathcal{C}) \rightarrow A(\mathcal{C})^* := \text{Hom}(A(\mathcal{C}), \mathbb{Z})$$

*be the map given by  $[W] \mapsto ([X] \mapsto \langle [W], [X] \rangle)$ . Let  $I$  be the set of shift orbits of isomorphism classes of objects in  $\mathcal{C}$ , and for each orbit  $\mathcal{O} \in I$  let  $A_{\mathcal{O}}$  be the span in  $A(\mathcal{C})$  of the  $[M]$  where  $M$  belongs to orbit  $\mathcal{O}$ . Then  $\phi(A(\mathcal{C}))$  is the direct sum  $\bigoplus_{\mathcal{O} \in I} \phi(A_{\mathcal{O}})$ . Furthermore, if  $\mathcal{O}$  is infinite or has odd length, then the restriction of  $\phi$  to  $A_{\mathcal{O}}$  is injective.*

*Proof.* As  $W$  and  $Z$  range through the shifts of some single fixed object the values of  $\langle W, \hat{Z} \rangle$  are the entries of the matrix

$$\begin{pmatrix} \ddots & & & \\ & 1 & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & \ddots \end{pmatrix}$$

when the orbit is infinite. The matrix is zero except on the leading diagonal and the diagonal immediately below. In the case of a finite shift orbit of length  $> 1$  the matrix is a circulant matrix with the same entries except for a 1 in the top right corner, and the matrix is (2) in the case of a shift orbit of length 1. If  $W$  and  $Z$  are not in the same shift orbit then  $\langle W, \hat{Z} \rangle = 0$ . The columns of these matrices give the values of the function  $\phi(W)$  on the  $\hat{Z}$ . This shows, first of all, that  $\phi$  sends objects from different shift orbits to independent functions. Furthermore, the matrix indicated has independent columns if either it is infinite, or if it is finite of odd size, so that in these cases the corresponding shift orbit is mapped injectively to  $A(\mathcal{C})^*$ .  $\square$

The last corollary is a statement about the non-degeneracy of the bilinear form we have constructed. We may reword it in the following more elementary fashion.

**Corollary 3.3.** *Let  $\mathcal{C}$  be a Krull-Schmidt  $k$ -linear triangulated category which is Hom-finite and which has Auslander-Reiten triangles. If  $W$  is an indecomposable object of  $\mathcal{C}$  then the values of  $\dim \text{Hom}(W, Z)$  as  $Z$  ranges over indecomposable objects determine the shift orbit to which  $W$  belongs. If the shift orbit containing  $W$  is either infinite or of odd length, then the isomorphism type of  $W$  is determined by the values of  $\dim \text{Hom}(W, Z)$ .*

This corollary may be compared with a result of Jensen, Su and Zimmermann [9, Prop. 4], who proved something similar without the hypothesis that  $\mathcal{C}$  should have Auslander-Reiten triangles, but with the additional requirement that objects  $W$  and  $W'$  which we are trying to distinguish should have  $\text{Hom}(W, W'[n]) = 0$  for some  $n$ . Their approach is based on an earlier result of Bongartz and is particularly appealing because it proceeds by elementary means. It is nevertheless interesting to see the connection between these results and the Auslander-Reiten triangles. For what it is worth, our result applies to categories such as cluster categories, or the stable module category of a group algebra, where the vanishing of a homomorphism space need not hold. In the case of the stable module category of a group algebra the phenomena which can occur were analyzed very fully by Benson and Parker in [6, Sec. 4]. For instance, their theory explains completely the following example. We state it without making reference to group algebras, but when  $k$  is a field of characteristic 5 the algebra in the example is isomorphic to the group algebra of a cyclic group of order 5.

**Example 3.4.** The stable module category for the ring  $k[X]/(X^5)$  has four indecomposable objects, namely the uniserial modules  $V_i = k[X]/(X^i)$  of dimension  $i$  where  $i = 1, 2, 3, 4$ . The dimensions of homomorphisms between these objects in the stable category are given in the following table.

	$V_1$	$V_2$	$V_3$	$V_4$
$V_1$	1	1	1	1
$V_2$	1	2	2	1
$V_3$	1	2	2	1
$V_4$	1	1	1	1

Modules  $V_1$  and  $V_4$  form an orbit of the shift operator, which is the inverse of the syzygy operator, and in this example they cannot be distinguished by dimensions of homomorphisms. The same is true of the modules  $V_2$  and  $V_3$ . We can, however, distinguish the shift orbits by means of dimensions of homomorphisms.

#### 4. THE SECOND GROTHENDIECK GROUP AND A HERMITIAN BILINEAR FORM

In the paper [6] of Benson and Parker it is shown that the almost split sequences give rise to elements of the Green ring which are dual to the standard basis of indecomposable modules with respect to the bilinear form given by dimensions of homomorphisms. We have seen in Section 3 that a similar statement is not immediately true for triangulated categories with Auslander-Reiten triangles, but that something close to this is true, in that the alternating sum of terms in an Auslander-Reiten triangle has non-zero product with two indecomposable objects, rather than just one.

We now show how to modify the bilinear form so that Auslander-Reiten triangles do indeed give dual elements to the standard basis. The approach requires us to modify the Grothendieck group as well. We will then show that for the category of perfect complexes over a symmetric algebra we obtain a Hermitian form, and when the algebra is a group algebra the form behaves well with respect to tensor product of complexes.

As before, let  $\mathcal{C}$  be a Krull-Schmidt  $k$ -linear triangulated category which is Hom-finite, and where  $k$  is algebraically closed. We define

$$A(\mathcal{C})^t := (\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} A(\mathcal{C}))/I$$

where  $t$  is an indeterminate and  $I$  is the  $\mathbb{Z}[t, t^{-1}]$ -submodule generated by expressions

$$1 \otimes [M[i]] - t^i \otimes [M]$$

for all objects  $M$  in  $\mathcal{C}$  and  $i \in \mathbb{Z}$ . We put

$$A_{\mathbb{Q}}(\mathcal{C})^t := \mathbb{Q}(t) \otimes_{\mathbb{Z}[t, t^{-1}]} A(\mathcal{C}).$$

It simplifies the notation to write  $M$  instead of  $[M]$  in  $A(\mathcal{C})^t$  and in  $A_{\mathbb{Q}}(\mathcal{C})^t$ . The tensor products are extension of scalars, and with this in mind we will write  $t^i M$  instead of  $t^i \otimes [M]$ . Thus in  $A(\mathcal{C})^t$  and in  $A_{\mathbb{Q}}(\mathcal{C})^t$  we have  $M[i] = t^i M$ , so that  $t$  acts as the shift operator.

**Proposition 4.1.** (1)  $A(\mathcal{C})^t \cong \bigoplus_{\text{shift orbits } \mathcal{O}} A_{\mathcal{O}}$  where  $A_{\mathcal{O}}$  is a free abelian group with basis the objects in orbit  $\mathcal{O}$ , regarded as a  $\mathbb{Z}[t, t^{-1}]$ -module via the action  $t^i M = M[i]$ . We understand that objects are taken up to isomorphism. If the shift orbit  $\mathcal{O}$  is infinite then  $A_{\mathcal{O}} \cong \mathbb{Z}[t, t^{-1}]$ , while if the shift operator has order  $n$  on  $\mathcal{O}$  then  $A_{\mathcal{O}} \cong \mathbb{Z}[t, t^{-1}]/(t^n - 1)$  as  $\mathbb{Z}[t, t^{-1}]$ -modules.  
(2) For each shift orbit  $\mathcal{O}$  (taken up to isomorphism) let  $M_{\mathcal{O}}$  be an object in  $\mathcal{O}$ . The  $M_{\mathcal{O}}$  where  $\mathcal{O}$  is infinite form a basis for  $A_{\mathbb{Q}}(\mathcal{C})^t$  over  $\mathbb{Q}(t)$ .

*Proof.* (1) We have  $A(\mathcal{C}) = \bigoplus_{\mathcal{O}} A_{\mathcal{O}}$  where  $A_{\mathcal{O}}$  is the span of the objects in orbit  $\mathcal{O}$ . Since each generator of  $I$  lies in some  $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} A_{\mathcal{O}}$  it follows that  $A(\mathcal{C})^t$  is the direct sum of the images of the  $\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} A_{\mathcal{O}}$  in it. Factoring out  $I$  has the effect of identifying each basis element  $M[i]$  of  $A_{\mathcal{O}}$  with  $t^i M$ . The identification of these spaces when the shift operator has infinite or finite order is immediate.

(2) On tensoring further with  $\mathbb{Q}(t)$  the summands  $\mathbb{Z}[t, t^{-1}]$  become copies of  $\mathbb{Q}(t)$ , and the summands  $\mathbb{Z}[t, t^{-1}]/(t^n - 1)$  become zero.  $\square$

On  $\mathbb{Q}(t)$  we denote by  $\overline{\phantom{x}}$  the field automorphism specified by  $t \mapsto t^{-1}$ . We will now assume that for every pair of objects  $M$  and  $N$  in  $\mathcal{C}$ ,  $\text{Hom}(M, N[i]) \neq 0$  for only finitely many  $i$ . We define a mapping

$$\langle \cdot, \cdot \rangle^t : A_{\mathbb{Q}}(\mathcal{C})^t \times A_{\mathbb{Q}}(\mathcal{C})^t \rightarrow \mathbb{Q}(t)$$

on basis elements  $M$  and  $N$  by

$$\begin{aligned} \langle M, N \rangle^t &:= \sum_{i \in \mathbb{Z}} t^i \dim \text{Hom}_{\mathcal{C}}(M, N[i]) \\ &= \sum_{i \in \mathbb{Z}} t^i \langle M, N[i] \rangle. \end{aligned}$$

We extend this definition to the whole of  $A_{\mathbb{Q}}(\mathcal{C})^t \times A_{\mathbb{Q}}(\mathcal{C})^t$  so as to have a sesquilinear form with respect to the field automorphism  $\overline{\phantom{x}}$ , that is,

$$\langle a_1 M_1 + a_2 M_2, N \rangle^t = a_1 \langle M_1, N \rangle^t + a_2 \langle M_2, N \rangle^t$$

and

$$\langle M, b_1 N_1 + b_2 N_2 \rangle^t = \overline{b_1} \langle M, N_1 \rangle^t + \overline{b_2} \langle M, N_2 \rangle^t$$

always hold.

We come to the main result of this section, which establishes the key properties of the form we have just defined, notably that Auslander-Reiten triangles give elements dual to the standard basis of indecomposables, and that for perfect complexes over a symmetric algebra the form is Hermitian.

**Theorem 4.2.** Let  $\mathcal{C}$  be a Krull-Schmidt  $k$ -linear triangulated category which is Hom-finite, and where  $k$  is algebraically closed. Suppose that  $\mathcal{C}$  satisfies the property that for every pair of objects  $M$  and  $N$  in  $\mathcal{C}$ ,  $\text{Hom}(M, N[i]) \neq 0$  for only finitely many  $i$ .

- (1) The expression defining  $\langle \cdot, \cdot \rangle^t$  gives a well-defined sesquilinear form on  $A_{\mathbb{Q}}(\mathcal{C})^t \times A_{\mathbb{Q}}(\mathcal{C})^t$ .
- (2) Whenever  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  is an Auslander-Reiten triangle in  $\mathcal{C}$  and  $\hat{Z} = X + Z - Y$  in  $A_{\mathbb{Q}}(\mathcal{C})^t$  we have

$$\langle M, \hat{Z} \rangle^t = \begin{cases} 0 & \text{unless } M \cong Z[i] \text{ for some } i, \\ (1+t) & \text{if } M \cong Z, \end{cases}$$

so that the element  $\frac{1}{1+t} \hat{Z}$  is dual on the right to  $Z$ . Similarly

$$\langle \hat{Z}, N \rangle^t = \begin{cases} 0 & \text{unless } N \cong X[i] \text{ for some } i, \\ (1+t^{-1}) & \text{if } N \cong X, \end{cases}$$

so that the element  $\frac{1}{1+t^{-1}} \hat{Z}$  is dual on the left to  $X$ .

- (3) When  $\mathcal{C} = D^b(\Lambda\text{-proj})$  is the category of perfect complexes for a symmetric algebra  $\Lambda$  the form is Hermitian, in the sense that  $\langle M, N \rangle^t = \overline{\langle N, M \rangle^t}$  always.
- (4) When  $\mathcal{C} = D^b(kG\text{-proj})$  is the category of perfect complexes for a group algebra  $kG$  we have

$$\langle M \otimes_k U, N \rangle^t = \langle M, U^* \otimes_k N \rangle^t$$

and

$$\langle M, N \rangle^t = \langle M^*, N^* \rangle^t.$$

*Proof.* (1) The expression which defines the form shows that

$$\begin{aligned} \langle M[j], N \rangle^t &= \sum_{i \in \mathbb{Z}} t^i \dim \text{Hom}_{\mathcal{C}}(M[j], N[i]) \\ &= \sum_{i \in \mathbb{Z}} t^{i+j} \dim \text{Hom}_{\mathcal{C}}(M[j], N[i+j]) \\ &= t^j \sum_{i \in \mathbb{Z}} t^i \dim \text{Hom}_{\mathcal{C}}(M, N[i]) \\ &= t^j \langle M, N \rangle^t \end{aligned}$$

and by a similar calculation  $\langle M, N[j] \rangle^t = t^{-j} \langle M, N \rangle^t$ . Thus the form vanishes when elements  $M[j] - t^j M$  are put in either side and consequently passes to a well defined sesquilinear form on  $A_{\mathbb{Q}}(\mathcal{C})^t$ .

(2) Because of the hypothesis that  $\text{Hom}(M, N[i]) \neq 0$  for only finitely many  $i$  we can never have  $M \cong M[1]$  for any  $M$ . We see by Proposition 3.1 that  $\langle M, \hat{Z} \rangle^t$  is only non-zero when  $M \cong Z[j]$  for some  $j$ , and that in the expression for  $\langle Z, \hat{Z} \rangle^t$  the only non-zero terms are  $\langle Z, \hat{Z} \rangle + t \langle Z, \hat{Z}[1] \rangle = 1+t$ . The argument for  $\langle \hat{Z}, N \rangle^t$  is similar.

(3) When  $\Lambda$  is symmetric the Nakayama functor is the identity. The identity functor on  $D^b(\Lambda\text{-proj})$  is a Serre duality, and we have  $\text{Hom}(M, N) \cong \text{Hom}(N, M)^*$ . Thus  $\langle M, N \rangle^t = \overline{\langle N, M \rangle^t}$  when  $M$  and  $N$  are basis elements

of  $A_{\mathbb{Q}}(\mathcal{C})^t$ , and the same formula follows for arbitrary elements of  $A_{\mathbb{Q}}(\mathcal{C})^t$  by the sesquilinear property of the form.

(4) The first two formulas follow from the identities  $\text{Hom}(M \otimes_k U, N) \cong \text{Hom}(M, U^* \otimes N)$  and  $\text{Hom}(M, N) \cong \text{Hom}(N^*, M^*)$ .  $\square$

It is interesting at this point to compare the bilinear form  $\langle M, N \rangle^t$  we have constructed to another bilinear form which appears in [5, page 13]. A bilinear form is defined there as

$$\langle M, N \rangle := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}(M, N[i])$$

which is the specialization of  $\langle M, N \rangle^t$  on putting  $t = -1$ . We see from Theorem 4.2 part (2) that this is exactly the specialization which destroys the possibility of having dual elements in our sense.

## 5. VALUES OF THE HERMITIAN FORM ON AUSLANDER-REITEN QUIVER COMPONENTS

We assume in this section that  $\Lambda$  is a symmetric algebra over  $k$  and we consider the homotopy category of perfect complexes  $D^b(\Lambda\text{-proj})$ . In this situation we know from [7] that  $D^b(\Lambda\text{-proj})$  has Auslander-Reiten triangles, and it is shown in [13] (see also [8]) that, provided  $\Lambda$  has no semisimple summand, all components of the Auslander-Reiten quiver of  $D^b(\Lambda\text{-proj})$  have the form  $\mathbb{Z}A_\infty$ . It follows from [7] (described also in [8]) that when  $\Lambda$  is a symmetric algebra the Auslander-Reiten triangles all have the form  $X \rightarrow Y \rightarrow X[1] \rightarrow X[1]$ , because the Nakayama functor is the identity. We say that a complex  $Z$  lies on the *rim* of the Auslander-Reiten quiver if, in the Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ , the complex  $Y$  is indecomposable.

We will show that the special shape of the Auslander-Reiten quiver and the fact that the bilinear form  $\langle \cdot, \cdot \rangle^t$  is Hermitian enable us to compute the values of the bilinear form given only its values on objects on the rim.

Assuming that  $\Lambda$  has no semisimple summand, we will label the objects in a component of the Auslander-Reiten quiver of  $D^b(\Lambda\text{-proj})$  as shown in Figure 1. Objects on the rim (the top row) are the shifts of a single object  $C_0$ , and at distance  $n$  from the rim the objects are shifts of an indecomposable  $C_n$ , which is chosen so that there is a chain of irreducible morphisms  $C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n$ .

In the next result we write  $\sigma_r := 1 + t + t^2 + \cdots + t^r \in \mathbb{Z}[t, t^{-1}]$  when  $r \geq 1$ , and  $\sigma_0 := 1$ .

**Theorem 5.1.** *Assume  $\Lambda$  is a symmetric  $k$ -algebra with no semisimple summand, and let  $C_0$  and  $D_0$  be indecomposable objects in  $D^b(\Lambda\text{-proj})$  which lie on the rim of their quiver components. Then the values of the form  $\langle C_m, D_n \rangle^t$  on objects in the same components as  $C_0$  and  $D_0$  are entirely determined by knowing  $\langle C_0, D_0 \rangle^t$ . Specifically, if  $C_0$  and  $D_0$  lie in different*

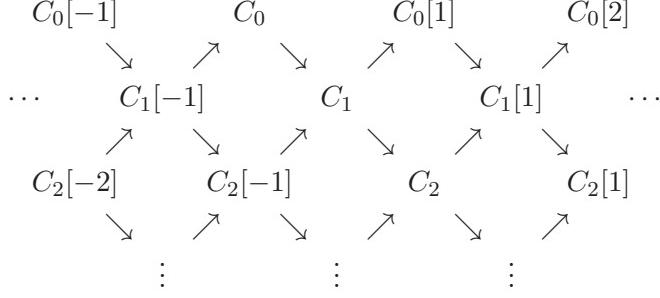


FIGURE 1. Auslander-Reiten quiver component of perfect complexes for a symmetric algebra

quiver components then

$$\langle C_m, D_n \rangle^t = \sigma_m \bar{\sigma}_n \langle C_0, D_0 \rangle^t$$

while

$$\langle C_m, C_n \rangle^t = \sigma_m \bar{\sigma}_n \left( \langle C_0, C_0 \rangle^t - \frac{(1+t)(1-t^\mu)}{1-t^{\mu+1}} \right)$$

where  $\mu$  is the maximum of  $m$  and  $n$ .

*Proof.* Step 1: we show that if  $C_m$  is not a shift of any  $D_i$  where  $0 \leq i \leq n-1$  then  $\langle C_m, D_n \rangle^t = \bar{\sigma}_n \langle C_m, D_0 \rangle^t$ . We proceed by induction on  $n$ . The result is true when  $n = 0$ . When  $n = 1$  the calculation is special because  $D_1$  is adjacent to the rim. Since  $C_m$  is not a shift of  $D_0$  we have

$$\begin{aligned} 0 &= \langle C_m, \hat{D}_0 \rangle^t \\ &= \langle C_m, D_0 \rangle^t + \langle C_m, D_0[-1] \rangle^t - \langle C_m, D_1[-1] \rangle^t \\ &= (1+t) \langle C_m, D_0 \rangle^t - t \langle C_m, D_1 \rangle^t. \end{aligned}$$

From this we deduce that

$$\langle C_m, D_1 \rangle^t = t^{-1}(1+t) \langle C_m, D_0 \rangle^t = \bar{\sigma}_1 \langle C_m, D_0 \rangle^t.$$

Now suppose that  $n \geq 2$  and the result holds for smaller values of  $n$ . We have

$$\begin{aligned} 0 &= \langle C_m, \hat{D}_{n-1} \rangle^t \\ &= \langle C_m, D_{n-1} \rangle^t + \langle C_m, D_{n-1}[-1] \rangle^t - \langle C_m, D_n[-1] \rangle^t - \langle C_m, D_{n-2} \rangle^t \\ &= (1+t) \langle C_m, D_{n-1} \rangle^t - t \langle C_m, D_n \rangle^t - \langle C_m, D_{n-2} \rangle^t. \end{aligned}$$

This is a recurrence relation for  $\langle C_m, D_n \rangle^t$  starting with the values already obtained when  $n = 0$  and  $1$ , and it is solved by

$$\langle C_m, D_n \rangle^t = \bar{\sigma}_n \langle C_m, D_0 \rangle^t.$$

Step 2: We deduce that if  $D_n$  is not a shift of any  $C_i$  where  $0 \leq i \leq m-1$  then  $\langle C_m, D_n \rangle^t = \sigma_m \langle C_0, D_n \rangle^t$ . This follows from Step 1 on exploiting the

fact that the form is Hermitian, for

$$\langle C_m, D_n \rangle^t = \overline{\langle D_n, C_m \rangle^t} = \overline{\sigma_m \langle D_n, C_0 \rangle^t} = \sigma_m \langle C_0, D_n \rangle^t.$$

Step 3: We put together Steps 1 and 2 to obtain the first statement of the Proposition, which applies when  $C_0$  and  $D_0$  lie in different quiver components.

Step 4: We treat the case of two objects in the same quiver component similarly, taking account of the fact that the values  $\langle C_m, \hat{C}_n \rangle^t$  are not always zero. First

$$\begin{aligned} 1 + t &= \langle C_0, \hat{C}_0 \rangle^t \\ &= \langle C_0, C_0 \rangle^t + \langle C_0, C_0[-1] \rangle^t - \langle C_0, C_1[-1] \rangle^t \\ &= (1 + t) \langle C_0, C_0 \rangle^t - t \langle C_0, C_1 \rangle^t \end{aligned}$$

so that

$$\langle C_0, C_1 \rangle^t = t^{-1}(1 + t)(\langle C_0, C_0 \rangle^t - 1) = \bar{\sigma}_1(\langle C_0, C_0 \rangle^t - 1)$$

which agrees with the formula we have to prove. By exactly the same calculation as was used in Step 1, taking  $n \geq 2$  and  $\hat{C}_{n-1} = \hat{D}_{n-1}$  and using the fact that  $0 = \langle C_0, \hat{C}_{n-1} \rangle^t$  we obtain the same recurrence

$$0 = (1 + t) \langle C_0, C_{n-1} \rangle^t - t \langle C_0, C_n \rangle^t - \langle C_0, C_{n-2} \rangle^t$$

valid when  $n \geq 2$ , and this has solution

$$\langle C_0, C_n \rangle^t = \bar{\sigma}_n \left( \langle C_0, C_0 \rangle^t - \frac{(1 + t)(1 - t^n)}{1 - t^{n+1}} \right).$$

Now if  $m \leq n$ , since  $C_m$  is not a shift of any  $C_i$  where  $0 \leq i \leq n-1$ , by Step 2 we deduce

$$\langle C_m, C_n \rangle^t = \sigma_m \bar{\sigma}_n \left( \langle C_0, C_0 \rangle^t - \frac{(1 + t)(1 - t^n)}{1 - t^{n+1}} \right).$$

Finally if  $m > n$  we use the Hermitian property to deduce

$$\begin{aligned} \langle C_m, C_n \rangle^t &= \overline{\langle C_n, C_m \rangle^t} \\ &= \sigma_n \bar{\sigma}_m \left( \langle C_0, C_0 \rangle^t - \frac{(1 + t)(1 - t^m)}{1 - t^{m+1}} \right) \\ &= \sigma_m \bar{\sigma}_n \left( \langle C_0, C_0 \rangle^t - \frac{(1 + t)(1 - t^m)}{1 - t^{m+1}} \right) \end{aligned}$$

which is what we have to prove.  $\square$

## 6. THE POSITION IN THE QUIVER OF PERFECT COMPLEXES WITH PRESCRIBED PROPERTIES

In this section we assume that  $\Lambda$  is a self-injective algebra over  $k$  (not necessarily a symmetric algebra). We will also suppose that  $\Lambda$  has no direct summand which is a semisimple algebra, so as before it follows from [13] that all components of the Auslander-Reiten quiver of  $D^b(\Lambda\text{-proj})$  have the

form  $\mathbb{Z}A_\infty$ . We will describe the position in the quiver of several kinds of indecomposable perfect complexes: those with one or two or (in the next section) three non-zero terms and those whose sequence of homology modules has a non-zero term sandwiched between two zero terms. This allows us to identify truncated projective resolutions as lying on the rim of the quiver, except in one specific case where they are adjacent to the rim. We do this by identifying aspects of the structure of complexes at a given distance from the rim: their length, and the composition factors of their homology modules. We will see that at a suitable distance from the rim the homology modules of the indecomposable complexes stabilize in a single module which we may call the *stabilization module*, and that provided  $\Lambda$  has radical length at least 3, this stabilization module can be made arbitrarily large. This latter result seems surprising, even for rings  $\Lambda$  such as  $k[X]/(X^3)$ . By comparison, indecomposable perfect complexes for  $k[X]/(X^2)$  cannot have homology modules of arbitrarily large dimension.

Before getting started on this we recall some basic facts about perfect complexes which are well known. They are proven (in the dual context of complexes of injectives) in [10, B2] except for the final statement, which is an easy exercise.

**Proposition 6.1.** *Every perfect complex over a finite dimensional algebra is homotopy equivalent to a complex whose terms all have minimal dimension among complexes in the same homotopy equivalence class. Such a minimal complex is unique up to isomorphism, and is a summand of every perfect complex homotopy equivalent to it. When the algebra is self-injective the minimal complex is characterized by the fact that every boundary map with non-zero target has kernel without non-zero projective summands, and it has the property that its top and bottom homology are non-zero.*

By the *length* of a perfect complex of  $\Lambda$ -modules we will mean the number of non-zero terms in the minimal complex chain homotopy equivalent to the given complex. Since  $\Lambda$  is supposed to be self-injective, this equals the difference between the degrees of the highest and lowest non-zero homology of the complex.

We will continue to use the fact, established in [7], that Auslander-Reiten triangles  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  have  $X[1] = \nu Z$  where  $\nu$  is the Nakayama functor. Over a self-injective algebra  $\Lambda$  the Nakayama functor is exact, since it is the composite of the Auslander-Bridger duality  $\text{Hom}_\Lambda(-, \Lambda)$  which is exact because  $\Lambda$  is injective, and the ordinary duality  $\text{Hom}_k(-, k)$  which is always exact. Thus  $\nu$  preserves the length of complexes, and  $H_n(X) = 0$  if and only if  $H_n(\nu X) = 0$ .

Let  $P_S$  be the projective cover of a simple  $\Lambda$ -module  $S$ . We start by describing part of the quiver component containing  $P_S$ , regarded as a complex whose only non-zero term is  $P_S$  in degree 0. The approach in the next proposition is by direct calculation, and it will be used later in Theorem 6.6 where we give complete information about this quiver component. For each

$\Lambda$ -module  $M$  we will denote by  $\mathcal{P}_M$  the complex  $P_1 \rightarrow P_0$  with terms in degrees 1 and 0 such that  $P_1 \rightarrow P_0 \rightarrow M$  is the start of a minimal projective resolution of  $M$ . Thus  $H_0(\mathcal{P}_M) \cong M$  and  $H_1(\mathcal{P}_M) \cong \Omega^2 M$ . We will be particularly interested in the complex  $\mathcal{P}_{\Omega^{-1}S}$ , which is the complex  $P_S \rightarrow \nu P_S$  where the map sends the simple top of  $P_S$  isomorphically to the socle of  $\nu P_S$  (= the injective hull of  $S$ ). We will also be interested in the complex  $\mathcal{H}_S$  which is defined to be  $\nu^{-1}P_S \rightarrow P_S \rightarrow \nu P_S$  in degrees 1, 0 and -1, where both maps send the top of one projective isomorphically to the socle of the next. It has homology  $\Omega \text{ Soc } P_S$ ,  $\text{Rad } P_S / \text{Soc } P_S$ ,  $\Omega^{-1}S$  in these degrees. We recall that the module  $H_S := \text{Rad } P_S / \text{Soc } P_S$  is known as the *heart* of  $P_S$ .

**Proposition 6.2.** *Let  $S$  be a simple module for a self-injective algebra  $\Lambda$  and let  $P_S$  be the projective cover of  $S$ . The Auslander-Reiten quiver of perfect complexes has the following shape close to  $P_S$ :*

$$\begin{array}{ccccc}
 \nu P_S[-1] & & P_S & & \nu^{-1}P_S[1] \\
 \searrow & \nearrow & \searrow & \nearrow & \\
 \cdots & \mathcal{P}_{\Omega^{-1}S}[-1] & \nu^{-1}\mathcal{P}_{\Omega^{-1}S} & \nu P_S[1] & \cdots \\
 \nearrow & \searrow & \nearrow & \searrow & \\
 & \mathcal{H}_S & & & \\
 & \vdots & & \vdots &
 \end{array}$$

Thus perfect complexes of length 1 lie on the rim of the Auslander-Reiten quiver, and the complexes of length 2 of the form  $\mathcal{P}_{\Omega^{-1}S} = (P_S \rightarrow \nu P_S)$  lie at distance 1 from the rim.

*Proof.* The construction of the Auslander-Reiten triangles comes from the description in [7]. Thus we have an Auslander-Reiten triangle  $\nu P_S[-1] \rightarrow \mathcal{P}_{\Omega^{-1}S}[-1] \rightarrow P_S \rightarrow \nu P_S$  where the final map sends the top of  $P_S$  to the socle of  $\nu P_S$  since this map is not homotopic to zero and lies in the socle of  $\text{Hom}(P_S, \nu P_S)$  and  $\mathcal{P}_{\Omega^{-1}S}$  is the mapping cone of this map. The complex  $\mathcal{P}_{\Omega^{-1}S}$  is indecomposable (or zero in case  $P_S = S$ ), because if it were not it would have to be the direct sum of two one-term complexes  $P_S$  and  $\nu P_S$  and would have projective homology. However, the homology of  $\mathcal{P}_{\Omega^{-1}S}$  is  $\Omega(S)$  and  $\Omega^{-1}(S)$  in degrees 1 and 0, and is not projective.

We construct the middle term of the Auslander-Reiten triangle starting at  $\mathcal{P}_{\Omega^{-1}S}[-1]$  as the mapping cone of a map  $\nu^{-1}\mathcal{P}_{\Omega^{-1}S}[-1] \rightarrow \mathcal{P}_{\Omega^{-1}S}[-1]$  which is not homotopic to zero and which lies in the socle of such maps under the action of the endomorphism ring of either complex. Such a map is

$$\begin{pmatrix} P_S \\ \alpha \uparrow \\ \nu^{-1}P_S \end{pmatrix} \xrightarrow{\nu(\alpha)} \begin{pmatrix} \nu P_S \\ \nu(\alpha) \uparrow \\ P_S \end{pmatrix} \xrightarrow{0} \begin{pmatrix} \nu P_S \\ \nu(\alpha) \uparrow \\ P_S \end{pmatrix}$$

and its mapping cone is a complex

$$\nu^{-1}P_S \xrightarrow{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} P_S \oplus P_S \xrightarrow{(\nu(\alpha), \nu(\alpha))} \nu P_S$$

The middle term has a direct summand  $\{(x, -x) \mid x \in P_S\} \cong P_S$  which is a direct complement to the subcomplex whose middle term is the first  $P_S$  direct summand. From this we see that the mapping cone is the direct sum as complexes  $P_S \oplus \mathcal{H}_S$ . This completes the description of this part of the Auslander-Reiten quiver.  $\square$

**Corollary 6.3.** *Let  $\Lambda$  be a self-injective algebra. Each quiver component of  $D^b(\Lambda\text{-proj})$  contains at most one indecomposable projective module regarded as a complex in degree 0.*

*Proof.* Any indecomposable projective module lies on the rim of its quiver component, which consists of that module and its shifts with the Nakayama functor applied, so no other projectives are possible.  $\square$

Assuming that the algebra  $\Lambda$  is self-injective and not necessarily symmetric, the complexes in a quiver component have the form shown in Figure 2, which is similar to Figure 1 but takes into account the non-identity Nakayama functor. We assume as always that  $\Lambda$  has no semisimple summand.

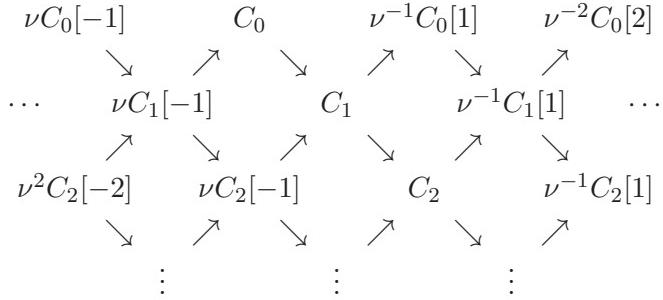


FIGURE 2. Auslander-Reiten quiver component of perfect complexes for a self-injective algebra

We will obtain a lot of information from the homology of complexes in a quiver component. Taking zero homology of every complex in the component we obtain modules

$$H_0(\nu^j C_i[-j]) \cong \nu^j H_j(C_i),$$

using the fact that  $\nu$  is exact. Thus every homology module of each complex  $C_0, C_1, \dots$  appears in the picture, up to a twist by the Nakayama functor, on taking zero homology in this fashion. This is shown in Figure 3, which we call the *homology diagram* of the quiver component.

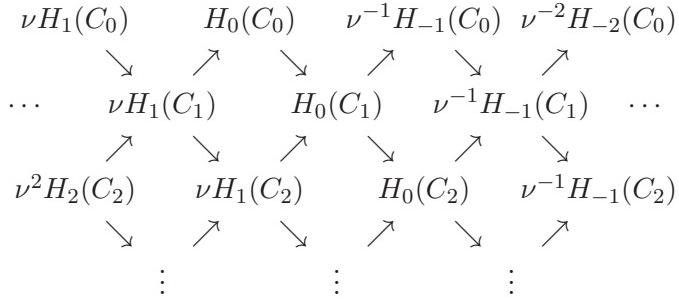


FIGURE 3. Homology diagram of an Auslander-Reiten quiver component of perfect complexes for a self-injective algebra

In the next results we will examine the pattern of modules in the homology diagram. We will do this both for quiver components which do and which do not contain projective modules. The basic idea is the same in both cases, but it is a little more complicated in the case of components which contain a projective. For reasons of clarity we consider quiver components which do not contain a projective module first.

For any object in a  $\mathbb{Z}A_\infty$  quiver component, by the *wing* of that object we mean the part of the quiver forming a triangle with the given object as vertex and a segment of the rim of the quiver as its opposite edge. We call that segment of the rim the *wing rim*, being the part of the rim which lies in the wing.

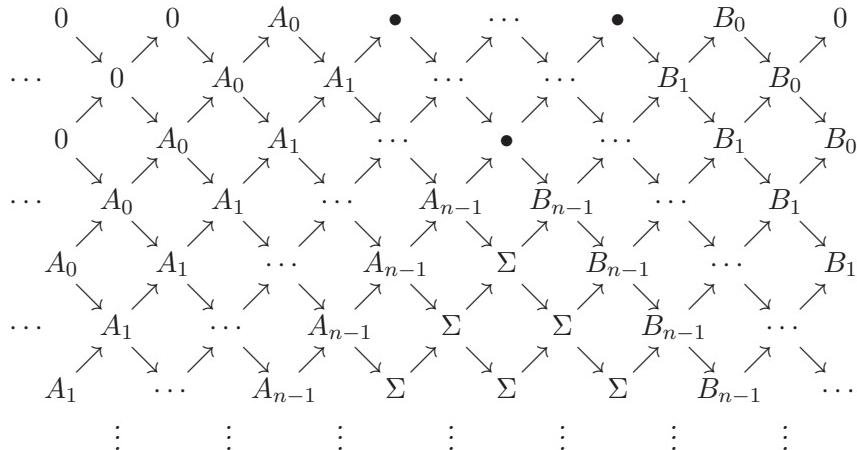
**Proposition 6.4.** *Let  $\Lambda$  be a self-injective algebra with no semisimple summand and consider the homology diagram of a quiver component of  $D^b(\Lambda\text{-proj})$  which does not contain a projective module. Every mesh in the diagram labels the modules in a short exact sequence of  $H_0$  modules. The composition factors (taken with multiplicity) of  $H_0$  of any complex are the union of the composition factors of the  $H_0$  of objects lying on the wing rim of the given object.*

*Proof.* Since the shift and the Nakayama functor are exact, it suffices to consider a mesh containing an object  $C_n$  at distance  $n$  from the rim in Figure 2. By Lemma 2.1, since homology is representable as  $\text{Hom}(\Lambda, -)$  and there is no projective module in the quiver component, its zero homology appears in a short exact sequence  $0 \rightarrow H_0(C_n) \rightarrow H_0(\nu^{-1}C_{n-1}[1]) \oplus H_0(C_{n+1}) \rightarrow H_0(\nu^{-1}C_n[1])$ , where if  $n = 0$  we take  $C_{-1} = 0$ . This establishes that each mesh corresponds to a short exact sequence.

To prove the statement about composition factors we proceed by induction on  $n$ . On the rim when  $n = 0$  the result holds. In case  $n = 1$  the exact sequence of  $H_0$  modules implies that the composition factors of the middle term are the union (with multiplicities) of the composition factors of the two outer terms, which is what we have to prove. When  $n \geq 2$  the composition factors of  $H_0(C_n)$  are those of the outer terms in the mesh above it, with

the composition factors of the top term in the mesh removed. By induction these are the composition factors from two parts of the rim (factors common to both parts taken twice) with a copy of the factors from the intersection of those parts of the rim removed. This gives exactly the desired result.  $\square$

**Theorem 6.5.** *Let  $\Lambda$  be a self-injective algebra and consider a quiver component of  $D^b(\Lambda\text{-proj})$  which does not contain a projective module. Adjacent to zeros on the rim of the homology diagram, this diagram has the shape shown below, with the maps between identically labelled terms being isomorphisms. Sufficiently far from the rim the homology modules stabilize to a module  $\Sigma$  as shown. Thus given two positions in the quiver for which the wing rims of the homology diagram have the same non-zero terms, the zero homology modules at those positions are isomorphic.*



*Proof.* We show that a zero on the rim of the homology diagram gives rise to homology groups in the pattern

$$(6.1) \quad \text{rim: } \begin{array}{ccccc} 0 & \xrightarrow{\quad} & A_0 & \xrightarrow{\quad} & \dots \\ & \searrow & \nearrow & \nearrow & \\ A_0 & \xrightarrow{\quad} & A_1 & \xrightarrow{\quad} & \\ & \searrow & \nearrow & \nearrow & \\ A_1 & \xrightarrow{\quad} & A_2 & \xrightarrow{\quad} & \end{array}$$

where each of the mappings  $A_i \rightarrow A_i$  is an isomorphism. Starting with a zero-homology module  $A_0$  on the rim which is adjacent to a zero homology module as shown, there is a short exact sequence of  $H_0$  modules of the Auslander-Reiten triangle at that point by Corollary 2.2:  $0 \rightarrow 0 \rightarrow U \rightarrow A_0 \rightarrow 0$  where initially  $U$  is unknown. From this sequence we deduce that the mapping  $U \rightarrow A_0$  is an isomorphism, which determines  $U$ . This fills in the term labelled  $A_0$  at distance 1 from the rim. Looking at the square of which this is the left-hand end and labelling the right-hand term  $A_1$ , we obtain a short exact sequence  $0 \rightarrow A_0 \rightarrow A_0 \oplus V \rightarrow A_1 \rightarrow 0$  for some unknown  $V$ , again from Corollary 2.2 applied to  $H_0$  of an Auslander-Reiten

triangle. The component map  $A_0 \rightarrow A_0$  is an isomorphism, so the sequence splits and we deduce that the component map  $V \rightarrow A_1$  is an isomorphism. This fills in  $V$  as  $A_1$ . Proceeding in this way we establish the pattern shown on the left side of diagram 6.1, and the pattern on the right adjacent to zeros on the rim of the homology diagram is established similarly.

The entries on the rim of the homology diagram are the various homology modules of a single perfect complex and its shifts, so they are eventually zero to the left and to the right. Sufficiently far from the rim the left and right patterns we have described overlap, and this happens in the region where we show modules  $\Sigma$ . At this point all morphisms in the diagram are isomorphisms, showing that the homology stabilizes. The positions labelled  $\Sigma$  are precisely those for which the wing rims in the homology diagram have the same non-zero terms, and this establishes the last statement of the theorem.  $\square$

We now describe the homology diagram of a quiver component containing a projective module and then use this to describe completely the entire quiver component.

**Theorem 6.6.** *Let  $S$  be a simple module for a self-injective algebra  $\Lambda$  and let  $P_S$  be the projective cover of  $S$ . We assume  $S \neq P_S$ .*

- (1) *The homology diagram of the quiver component containing  $P_S$  is as follows:*

$$\begin{array}{ccccccc}
& 0 & & 0 & \times & P_S & \times & 0 & 0 \\
& \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
\cdots & 0 & & \text{Rad } P_S & & P_S / \text{Soc } P_S & & 0 & \cdots \\
& \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\
& 0 & & \text{Rad } P_S & & H_S & & P_S / \text{Soc } P_S & 0 \\
& \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
\cdots & \text{Rad } P_S & & H_S & & H_S & & P_S / \text{Soc } P_S & \cdots \\
& \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow \\
& \text{Rad } P_S & & H_S & & H_S & & H_S & P_S / \text{Soc } P_S \\
& \vdots & & \vdots & & \vdots & & \vdots & \vdots
\end{array}$$

Here the module  $H_S = \text{Rad } P_S / \text{Soc } P_S$  is the heart of  $P_S$ . Every mesh in the diagram labels the terms in a short exact sequence of zero homology modules, except for the two meshes labelled  $\times$ . In every case the short exact sequence is split except for the one underneath  $P_S$ , which is an Auslander-Reiten sequence. Sufficiently far from the rim the zero homology stabilizes to be  $H_S$ .

- (2) *With the notation of Figure 2 the complex  $C_n$  at distance  $n$  from the rim of the quiver component containing  $P_S$  is*

$$\nu^{-n} P_S \rightarrow \cdots \rightarrow \nu^{-2} P_S \rightarrow \nu^{-1} P_S \rightarrow P_S$$

where each map sends the simple top of a projective module isomorphically to the socle of the next module. The irreducible morphisms between these complexes and their Auslander-Reiten translates are represented by the obvious inclusions and surjections between such complexes of adjacent lengths.

*Proof.* (1) The fact that all meshes correspond to short exact sequences except the ones marked  $\times$  follows from Lemma 2.1 and the description of the quiver component given in Proposition 6.2, as in the proof of Proposition 6.4. The zero homology of the complexes shown in Proposition 6.2 has already been computed before that proposition, and accounts for terms on the rim and the mesh below  $P_S$ . The facts that the remaining meshes correspond to split short exact sequence with maps between identically labelled terms being isomorphisms, and that the homology stabilizes at  $H_S$ , follow by the same argument as in Theorem 6.5. We may deduce that the mesh below  $P_S$  corresponds to an Auslander-Reiten sequence by observing from our earlier calculation that the component maps to  $P_S$  and  $H$  in the mesh are mono and epi, while the component maps in the mesh from  $P_S$  and  $H$  are epi and mono. This identifies the sequence as a well-known Auslander-Reiten sequence [3, Sect. 4.11]. We will also see in Proposition 7.2 by a different argument that the short exact sequence under  $P_S$  is an Auslander-Reiten sequence.

(2) We show that  $C_n$  can be identified as stated by induction on  $n$ . We see from Proposition 6.2 that the expressions for  $C_0$ ,  $C_1$  and  $C_2$  are correct. Now suppose that  $n \geq 2$ , that the result is true for  $n$  and smaller values, and consider  $C_{n+1}$ . From the general structure of the quiver shown in Figure 2 the mapping cone of  $\nu^{-1}C_n \rightarrow C_n$  is  $C_{n+1} \oplus \nu^{-1}C_{n-1}[1]$ . The mapping cone has the form

$$\nu^{-n-1}P_S \rightarrow (\nu^{-n}P_S)^2 \rightarrow \cdots \rightarrow (\nu^{-2}P_S)^2 \rightarrow (\nu^{-1}P_S)^2 \rightarrow P_S$$

and  $\nu^{-1}C_{n-1}[1]$  has the form

$$\nu^{-n}P_S \rightarrow \cdots \rightarrow \nu^{-2}P_S \rightarrow \nu^{-1}P_S$$

so that

$$C_{n+1} \cong \nu^{-n-1}P_S \rightarrow \nu^{-n}P_S \rightarrow \cdots \rightarrow \nu^{-2}P_S \rightarrow \nu^{-1}P_S \rightarrow P_S.$$

From the homology diagram in part (1) we see that the homology modules of  $C_{n+1}$  are  $\nu^{-n-1}\text{Rad } P_S$ ,  $\nu^{-n}H_S, \dots, \nu^{-1}H_S, P_S/\text{Soc } P_S$  and by a dimension count it follows that each map in  $C_{n+1}$  sends the simple top of each projective to the socle of the next.  $\square$

We deduce more corollaries from these patterns. Throughout we assume that  $\Lambda$  is a self-injective algebra with no semisimple summand and we consider the Auslander-Reiten quiver of perfect complexes.

**Corollary 6.7.** *If complexes on the rim of a quiver component have length  $t$  then complexes at distance  $r$  from the rim have length  $t+r$ .*

*Proof.* This follows from Theorems 6.5 and 6.6, since the length of a complex is determined from its extreme non-vanishing homology modules, and so equals the length of the part of the row in the homology diagram where the terms are non-zero. These theorems show that this length grows by 1 with each step away from the rim.  $\square$

**Corollary 6.8.** *Any (indecomposable) perfect complex with two non-zero terms, not of the form  $P_S \rightarrow \nu P_S$  for some simple module  $S$  (where the map sends the top of  $P_S$  isomorphically to the socle of  $\nu P_S$ ), lies on the rim of its quiver component.*

*Proof.* If it didn't lie on the rim, the complexes on the rim of its quiver component would have to have length 1 by Corollary 6.7, so would be shifts of a projective module  $P_S$  for some simple module  $S$ . In that case the 2-term complexes next to the rim have the form which has been excluded.  $\square$

**Corollary 6.9.** *In an indecomposable complex at distance  $n$  from the rim, any non-zero homology module must occur as part of a string of at least  $n+1$  non-zero consecutive homology modules.*

*Proof.* The result is true for quiver components containing a projective module by Theorem 6.6. For other components, non-zero homology comes about from non-zero homology on the rim, by Proposition 6.4. Each non-zero term on the rim gives rise to  $n+1$  non-zero consecutive homology modules at distance  $n$ . This is the only way we get non-zero homology at distance  $n$  from the rim and so every non-zero homology module must be part of a string of  $n+1$  non-zero homology modules.  $\square$

**Corollary 6.10.** *The only perfect complexes of length  $n+1$  which are at distance  $n$  from the rim of their quiver component are the complexes*

$$\nu^{-n}P_S \rightarrow \cdots \rightarrow \nu^{-2}P_S \rightarrow \nu^{-1}P_S \rightarrow P_S$$

*for some simple module  $S$ , where every mapping sends a simple top isomorphically to the next simple socle. All other complexes of length  $n+1$  are closer to the rim than this.*

*Proof.* By Corollary 6.7 all complexes of length  $n+1$  must lie at distance  $n$  or less from the rim. If such a complex lies at distance  $n$  then by the same corollary the complexes on the rim of the quiver component have length 1, and so are projective modules  $P_S$  in a single degree. We saw in Theorem 6.6 that the complexes at distance  $n$  from the rim in such a quiver component have the form stated.  $\square$

**Corollary 6.11.** *Any indecomposable complex  $X$  with a sequence of homology modules of the form  $H_{d+1}(X) = 0$ ,  $H_d(X) \neq 0$  and  $H_{d-1}(X) = 0$  must lie on the rim of its quiver component.*

*Proof.* If  $X$  were not on the rim, non-zero homology strings would all have to have length at least 2 by Corollary 6.9. Since the complex has a non-zero homology string of length 1, it must lie on the rim.  $\square$

**Proposition 6.12.** *Let  $M$  be an indecomposable module, and let  $P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$  be the start of a minimal resolution of  $M$ . Then the complex  $P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$  is indecomposable. It lies on the rim of its quiver component unless  $M = \Omega^{-1}S$  for some simple module  $S$  and  $n = 1$ , in which case the complex lies at distance 1 from the rim of its quiver component.*

*Proof.* If the complex were to decompose as a direct sum of two complexes, one of the summands would have  $M$  as its zero homology. That summand would have projective terms, and would be the start of a smaller projective resolution of  $M$ . Since we chose a minimal projective resolution, this cannot happen.

If  $n > 1$  then the homology of the complex has its zero homology group isolated in the sense of Corollary 6.11 and so the complex lies on the rim of its quiver component. The other possibility is that  $n \leq 1$  so that the complex has one or two terms. If it has one term ( $n = 0$ ) then  $M$  is projective and lies on the rim of the quiver by Proposition 6.2. If  $n = 1$  then by hypothesis the complex is not isomorphic to  $P_S \rightarrow \nu P_S$  and so lies on the rim by Corollary 6.8.  $\square$

The next result has some independent significance, in that it makes no reference to the Auslander-Reiten quiver.

**Theorem 6.13.** *Let  $\Lambda$  be a self-injective algebra of radical length at least 3. There exist indecomposable perfect complexes with degree zero homology of arbitrarily large dimension.*

*Proof.* We may construct a perfect complex with arbitrarily many non-zero homology modules, and for some  $H \neq 0$  having terms  $0, H, 0$  somewhere in the list of homology groups. For example, we may take an indecomposable projective module  $P_S$  of radical length at least 3, form a complex

$$P_{\text{Rad } P_S} \rightarrow P_S \rightarrow \nu P_S \rightarrow \nu^2 P_S \rightarrow \cdots \rightarrow$$

where the first map is the projective cover of the radical of  $P_S$ , and after that each map identifies the top of a module with the socle of the next. Such a complex can be made to have arbitrarily many non-zero homology modules, and the second homology module from the left is zero. Such a complex must lie on the rim of its quiver component by Corollary 6.11. Considering homology at some distance from the rim the stabilizing module  $\Sigma$  which appears in Corollary 6.5 has composition length which is the sum of the lengths of all homology modules of the starting complex. We conclude that this may be made arbitrarily large.  $\square$

## 7. PERFECT COMPLEXES WITH PRESCRIBED HOMOLOGY AND AUSLANDER-REITEN SEQUENCES

We start with a result which makes a connection between Auslander-Reiten sequences and Auslander-Reiten triangles. It turns out to be a well

known result in the context of modules, expressed in the language of complexes. For this result  $\Lambda$  can be any finite dimensional algebra, not necessarily one that is self-injective.

**Proposition 7.1.** *Let  $M$  be a non-projective indecomposable  $\Lambda$ -module and let  $\mathcal{P}_M = (P_1 \rightarrow P_0)$  be the first two terms of a minimal projective resolution of  $M$ , so that  $H_0(\mathcal{P}_M) \cong M$  and  $H_1(\mathcal{P}_M) \cong \Omega^2 M$ . Consider the Auslander-Reiten triangle  $\nu\mathcal{P}_M[-1] \rightarrow \mathcal{E}_M \rightarrow \mathcal{P}_M \rightarrow \nu\mathcal{P}_M$  where  $\nu$  is the Nakayama functor. Then the part of the long exact sequence in homology of this triangle, whose terms are the zero homology groups, is the Auslander-Reiten sequence of  $\Lambda$ -modules  $0 \rightarrow \tau M \rightarrow E_M \rightarrow M \rightarrow 0$ . The non-zero homology groups of the mapping cone complex  $\mathcal{E}_M$  are  $H_1(\mathcal{E}_M) = \Omega^2 M$ ,  $H_0(\mathcal{E}_M) = E_M$  and  $H_{-1}(\mathcal{E}_M) = \nu M$ .*

*Proof.* We know from [2, IV.2.4] that  $\nu\mathcal{P}_M[-1]$  has homology modules

$$H_{-1}(\mathcal{P}_M[-1]) = \nu M \quad \text{and} \quad H_0(\mathcal{P}_M[-1]) = \tau M.$$

By Corollary 2.2, since  $M$  is not projective, the long exact sequence in homology of the triangle breaks up as a splice of short exact sequences, of which the 0-homology sequence has the form

$$0 \rightarrow \tau M \rightarrow H_0(\mathcal{E}_M) \rightarrow M \rightarrow 0.$$

This sequence has the correct end terms to be an Auslander-Reiten sequence. We show first that the sequence has the lifting property of Auslander-Reiten sequences, and then later that it is not split. This will establish that it is indeed an Auslander-Reiten sequence.

Suppose that we have an indecomposable module  $N$  and a morphism  $N \rightarrow M$  which is not an isomorphism. Take the start of a minimal projective resolution  $\mathcal{Q} = (Q_1 \rightarrow Q_0)$  of  $N$  and lift the morphism  $N \rightarrow M$  to a morphism of complexes  $\mathcal{Q} \rightarrow \mathcal{P}_M$ . This mapping is not split epi, because on taking zero homology it is not split epi. It therefore lifts to a morphism  $\mathcal{Q} \rightarrow \mathcal{E}_M$  and on taking zero homology we deduce that the original morphism lifts to a morphism  $N \rightarrow H_0(\mathcal{E}_M)$ .

To show that the sequence of zero homology groups is not split, suppose to the contrary that we have a splitting map  $M \rightarrow H_0(\mathcal{E}_M) = Z_0(\mathcal{E}_M)/B_0(\mathcal{E}_M)$  with image  $U/B_0(\mathcal{E}_M)$  for some submodule  $U \subseteq Z_0(\mathcal{E}_M)$ . By the projective property of  $P_0$  this lifts to a map  $P_0 \rightarrow U$  and hence also to a map  $P_1 \rightarrow (\mathcal{E}_M)_1$ , so that we have a chain map  $\mathcal{P}_M \rightarrow \mathcal{E}_M$ . Composing with the map  $\mathcal{E}_M \rightarrow \mathcal{P}_M$  we obtain an endomorphism of  $\mathcal{P}_M$  which lifts the identity on  $M$ . By Fitting's Lemma some power of this endomorphism has image a summand of  $\mathcal{P}_M$ , which again must be the start of a resolution of  $M$ . By minimality of  $\mathcal{P}_M$  this summand must be the whole of  $\mathcal{P}_M$ , and so the endomorphism of  $\mathcal{P}_M$  is an isomorphism and we have split the Auslander-Reiten triangle – a contradiction.

We have now shown that taking zero homology gives an Auslander-Reiten sequence. To identify the top and bottom homology groups of  $\mathcal{E}_M$  we use

the fact again from Corollary 2.2 that the degree 1 and degree  $-1$  terms in the long exact homology sequence give short exact sequences, and they are

$$0 \rightarrow 0 \rightarrow H_1(\mathcal{E}_M) \rightarrow H_1(\mathcal{P}_M) = \Omega^2 M \rightarrow 0$$

and

$$0 \rightarrow H_0(\nu\mathcal{P}_M) = \nu M \rightarrow H_{-1}(\mathcal{E}_M) \rightarrow 0 \rightarrow 0.$$

This completes the proof.  $\square$

We comment that the construction we have just described gives a way to implement the construction of Auslander-Reiten sequences on a computer, without explicitly computing Ext groups and extensions corresponding to Ext classes (although the ingredients in the construction are closely related to this). Note also that we do not assume that  $\Lambda$  is self-injective here, so the algorithm works for any finite dimensional algebra. The algorithm is as follows, and relies on the construction of Auslander-Reiten triangles given in [7]:

- (1) Given a module  $M$ , compute the first two terms  $\mathcal{P}_M = (P_1 \rightarrow P_0)$  in a minimal projective resolution of  $M$ .
- (2) Compute the effect of the Nakayama functor  $\nu\mathcal{P}_M$ . (For some algebras this is particularly easy. For example, with symmetric algebras the Nakayama functor is the identity.)
- (3) Compute a chain map  $\mathcal{P}_M \rightarrow \nu\mathcal{P}_M$  which is not homotopic to zero, and which is annihilated by  $\text{Rad}(\text{End}(\mathcal{P}_M))$ . This can be realized always as a chain map which sends  $P_0$  to the socle of  $\nu P_0$  and which sends  $P_1$  to 0. The map  $P_0 \rightarrow \text{Soc } \nu P_0$  must vanish on the image of  $\text{Rad } \text{End}(M)$  in  $M/\text{Rad } M \cong P_0/\text{Rad } P_0$ .
- (4) Compute the mapping cone  $\mathcal{E}_M[1]$  of the map in (3).
- (5) Compute the zero homology of the sequence  $\nu\mathcal{P}_M[-1] \rightarrow \mathcal{E}_M \rightarrow \mathcal{P}_M$ .

The above algorithm has similar ingredients in it to the more obvious approach of computing  $\tau M$  as  $H_1(\nu\mathcal{P}_M)$ , computing  $\text{Ext}^1(M, \tau M)$ , finding an element in the socle under the action of  $\text{End } M$  and then constructing the corresponding extension as a pushout, but some parts of this are streamlined, in that we do not have to compute the whole Ext group, and the construction of the resulting extension is done automatically within the mapping cone construction.

It is intriguing that when  $\Lambda$  is a symmetric algebra, the homology modules of the 3-term complex  $\mathcal{E}_M$  we have just constructed are the modules which appear in an Auslander-Reiten sequence, as we observe in the next corollary.

**Corollary 7.2.** *If  $\Lambda$  is a symmetric algebra and  $M$  is a non-projective indecomposable  $\Lambda$ -module, there is a 3-term perfect complex  $\mathcal{E}_M$  with  $H_1(\mathcal{E}_M) = \tau M$ ,  $H_0(\mathcal{E}_M) = E$  and  $H_{-1}(\mathcal{E}_M) = M$ , where  $0 \rightarrow \tau M \rightarrow E_M \rightarrow M \rightarrow 0$  is the Auslander-Reiten sequence of  $\Lambda$ -modules terminating at  $M$*

*Proof.* When  $\Lambda$  is symmetric the Nakayama functor is the identity and  $\tau M \cong \Omega^2 M$ . Thus the complex  $\mathcal{E}_M$  of Proposition 7.2 is perfect and has the desired homology.  $\square$

It is not always possible to construct perfect complexes with prescribed homology groups. One constraint on the homology groups is that their alternating sum must lie in the span of projective modules in the appropriate Grothendieck group. Even when this condition is satisfied, it is still not always possible to realize a list of groups as homology groups, and we mention in this context the result of Carlsson and Allday-Puppe, for which a different approach has been given recently by Avramov, Buchweitz, Iyengar and Miller [4]: if  $\mathcal{P}$  is a perfect complex of  $kG$ -modules for an elementary abelian  $p$ -group  $G = C_p^r$ , where  $k$  is a field of characteristic  $p$ , then

$$\sum_i \text{Loewy length } H_i(\mathcal{P}) \geq r + 1.$$

This non-existence result complements the content of Corollary 7.2 and allows us to see that even with three modules which appear in a short exact sequence, it is not always possible to realize those modules as the homology of a perfect complex of length 3.

We can now characterize the positions of complexes of length 3 in the Auslander-Reiten quiver.

**Proposition 7.3.** *Let  $\Lambda$  be a self-injective algebra,  $M$  an indecomposable  $\Lambda$ -module and  $\mathcal{E}_M$  the 3-term complex which is the mapping cone constructed above.*

- (1) *The complex  $\mathcal{E}_M$  is indecomposable unless  $M = \Omega^{-1}S$  for some simple module  $S$ , in which case  $\mathcal{E}_M$  is the direct sum of an indecomposable 3-term complex and  $P_S$  as a complex in degree 0.*
- (2) *An indecomposable 3-term perfect complex lies on the rim of its quiver component unless, up to shift, it is either of the form  $\mathcal{E}_M$  for some  $M \not\cong \Omega^{-1}S$  (in which case it lies at distance 1 from the rim) or it is the indecomposable 3-term summand of  $\mathcal{E}_{\Omega^{-1}S}$  for some simple  $\Lambda$ -module  $S$  (in which case it lies at distance 2 from the rim).*

*Proof.* (1) The complexes  $\mathcal{P}_M$  lie on the rim of the quiver by Corollary 6.8 unless  $M = \Omega^{-1}S$  for some simple module  $S$ . Apart from this exception,  $\mathcal{E}_M$  is the middle term of the Auslander-Reiten triangle with third term  $\mathcal{P}_M$ , and it is indecomposable since  $\mathcal{P}_M$  lies on the rim. When  $M = \Omega^{-1}S$  we have seen the decomposition of  $\mathcal{E}_M$  in Theorem 6.6.

(2) We have seen from Corollary 6.10 that indecomposable 3-term complexes lie at distance at most 2 from the rim, and the ones at distance 2 have the form  $\nu^{-1}P_S \rightarrow P_S \rightarrow \nu P_S$ . If a 3-term complex lies at distance 1 from the rim, the complexes on the rim must have length 2, and are (up to shift) of the form  $\mathcal{P}_M$  with  $M \not\cong \Omega^{-1}S$  for any simple  $S$ , by Corollary 6.8. Apart from these cases, 3-term complexes must lie on the rim.  $\square$

Thus, if a 3-term perfect complex has homology modules which are not the terms in an Auslander-Reiten sequence of  $A$ -modules (to within a projective summand of the middle term) then the complex lies on the rim of the Auslander-Reiten quiver of perfect complexes.

## 8. THE POSITION OF RIGID COMPLEXES IN THE AUSLANDER-REITEN QUIVER

We will show that over a symmetric algebra an indecomposable complex  $C$  which is *rigid*, i.e.  $\text{Hom}(C, C[1]) = 0$ , must lie on the rim of the quiver. It follows that summands of tilting complexes lie on the rim, but this could be deduced from Rickard's theory of tilting complexes, since they a tilting complex is the image of the regular representation under a derived equivalence, and we have already seen that projective modules lie in the rim.

We will deduce our result from the following lemma on Auslander-Reiten triangles, whose analogue for Auslander-Reiten sequences is also true by the same argument.

**Lemma 8.1.** *Let  $A \rightarrow B \rightarrow C \rightarrow A[1]$  be an Auslander-Reiten triangle in a Krull-Schmidt triangulated category  $\mathcal{C}$  and suppose that  $\text{Hom}(A, C) = 0$ . Then  $A$  and  $C$  lie on the rim of the Auslander-Reiten quiver of  $\mathcal{C}$ .*

*Proof.* Suppose to the contrary that  $B = B_1 \oplus B_2$ . We show that it is not possible for both of the composite maps  $A \rightarrow B_1 \rightarrow C[1]$  and  $A \rightarrow B_2 \rightarrow C[1]$  to be zero as follows. Let  $W$  be any object and consider the long exact sequence

$$\cdots \longrightarrow \text{Hom}(W, C[-1]) \xrightarrow{\quad} \cdots$$

If both composites  $A \rightarrow B_i \rightarrow C$  were zero for  $i = 1, 2$ , then both composites  $\text{Hom}(W, C) \rightarrow \text{Hom}(W, B_i) \rightarrow \text{Hom}(W, C)$  would be zero. It would follow that both projections of  $\text{Ker } \beta = \text{Im } \alpha$  onto the two summands  $\text{Hom}(W, B_i)$  lie in  $\text{Ker } \beta$  and hence

$$\alpha \text{Hom}(W, A) = (\text{Ker } \beta \cap \text{Hom}(W, B_1)) \oplus (\text{Ker } \beta \cap \text{Hom}(W, B_2))$$

is a direct sum of  $\text{End}(W)$ -modules. Take  $W = A$ . Because  $\text{End}(A)$  is local we deduce that  $\alpha \text{Hom}(A, A)$  has a unique simple quotient as an  $\text{End } A$ -module. From this it follows that one of the summands  $\text{Ker } \beta \cap \text{Hom}(A, B_i)$  is zero and that  $\alpha \text{Hom}(A, A) \subseteq \text{Hom}(A, B_j)$  for the other suffix  $j$ . Now  $\alpha(1_A)$  is represented by the two component maps  $A \rightarrow B_1$  and  $A \rightarrow B_2$ , so one of these must be zero. However, these component maps are irreducible morphisms and are never zero. This contradiction shows that one of the

composite maps  $A \rightarrow B_1 \rightarrow C$  and  $A \rightarrow B_2 \rightarrow C$  is non-zero (and hence both are). We complete the proof by invoking the hypothesis that there is no non-zero map  $A \rightarrow C$ . This further contradiction establishes that  $B$  is indecomposable.  $\square$

We can now deduce our application to rigid complexes.

**Theorem 8.2.** *Let  $C$  be an indecomposable perfect complex for a symmetric algebra  $\Lambda$  with the property that  $\text{Hom}(C, C[1]) = 0$ . Then  $C$  lies on the rim of its component in the Auslander-Reiten quiver.*

*Proof.* The Auslander-Reiten triangle starting at  $C$  has the form  $C \rightarrow X \rightarrow C[1] \rightarrow C[1]$  since the Nakayama functor is the identity. Since  $\text{Hom}(C, C[1]) = 0$ , the result follows from Lemma 8.1.  $\square$

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